SECOND ORDER ESTIMATES FOR HESSIAN EQUATIONS OF PARABOLIC TYPE ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we establish the second order estimates of solutions to the first initial-boundary value problem for general Hessian type fully nonlinear parabolic equations on Riemannian manifolds. The techniques used in this article can work for a wide range of fully nonlinear PDEs under very general conditions.

Keywords: Fully nonlinear parabolic equations, Riemannian manifolds, *a priori* estimates, The first initial-boundary value problem.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M and $\bar{M} := M \cup \partial M$. We will study the equation

$$f(\lambda(\nabla^2 u + A[u])) - u_t = \psi(x, t, u, \nabla u) \tag{1.1}$$

in $M_T = M \times (0, T] \subset M \times \mathbb{R}$, where f is a symmetric smooth function of n variables, $\nabla^2 u$ denotes the Hessian of $u(x, t)$ with respect to $x \in M$, $A[u] = A(x, t, \nabla u)$ is a $(0, 2)$ tensor on $\bar{M}$ which may depend on $t \in [0, T]$ and ∇u and

$$\lambda(\nabla^2 u + A[u]) = (\lambda_1, \ldots, \lambda_n)$$

denotes the eigenvalues of $\nabla^2 u + A[u]$ with respect to the metric g.

In this paper we are mainly concerned with the *a priori* C^2 estimates for solutions to (1.1) with boundary condition

$$u = \varphi \text{ on } \mathcal{P}M_T, \tag{1.2}$$

where $\varphi \in C^\infty(\overline{\mathcal{P}M_T})$ satisfying $\lambda(\nabla^2 \varphi(x, 0) + A[\varphi(x, 0)]) \in \Gamma$ for all $x \in \bar{M}$. Here $\mathcal{P}M_T = BM_T \cup SM_T$ is the parabolic boundary of M_T with $BM_T = M \times \{0\}$ and $SM_T = \partial M \times [0, T]$.

The idea of this paper is mainly from Guan and Jiao [7] where the authors studied the second order estimates for the elliptic counterpart of (1.1):

$$f(\lambda(\nabla^2 u + A(x, u, \nabla u))) = \psi(x, u, \nabla u). \tag{1.3}$$

Comparing with the elliptic case, the main difficulty in deriving the second order estimates for the parabolic equation (1.1) is from its degeneracy which is overcome by using the strict subsolution in this paper. Surprisingly, thanks to the strict subsolution, we are able to relax some restrictions to f. Again because of the degeneracy,

we do not get the higher estimates and the existence of classical solution. It is useful to consider viscosity solutions to (1.1) which will be addressed in forthcoming papers.

The first initial-boundary value problem for equation of form (1.1) in $\mathbb{R}^n$ with $A \equiv 0$ and $\psi = \psi(x,t)$ was studied by Ivochkina and Ladyzhenskaya in [8] (when $f = \sigma_n^{1/n}$) and [9]. Jiao and Sui treated the case that $A \equiv \chi(x)$ and $\psi = \psi(x,t)$ on Riemannian manifolds using the techniques of [5] and [7]. For the elliptic Hessian equations on manifolds, we refer the readers to Li [11], Urbas [13], Guan [4, 5, 6], Guan and Jiao [7] and their references.

As in [2], in which the authors studied the equations (1.3) with $A \equiv 0$ and $\psi = \psi(x)$ in a bounded domain of $\mathbb{R}^n$, $f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ is assumed to be defined on Γ, where Γ is an open, convex, symmetric proper subcone of $\mathbb{R}^n$ with vertex at the origin and

$$\Gamma^+ \equiv \{\lambda \in \mathbb{R}^n : \text{ each component } \lambda_i > 0\} \subseteq \Gamma,$$

and to satisfy the following structure conditions in this paper:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n, \tag{1.4}$$

$$f \text{ is concave in } \Gamma, \tag{1.5}$$

and

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial\Gamma. \tag{1.6}$$

Typical examples are given by $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n$, defined in the cone $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k\}$, where $\sigma_k(\lambda)$ are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n.$$

Another interesting example is $f = \log P_k$, where

$$P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \leq k \leq n,$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0\}.$$

We call a function $u(x,t)$ admissible if $\lambda(\nabla^2 u + A[u]) \in \Gamma$ in $M \times [0,T]$. It is shown in [2] that (1.4) ensures that equation (1.1) is parabolic for admissible solutions. (1.5) means that the function F defined by $F(A) = f(\lambda[A])$ is concave for $A \in \mathcal{S}^{n\times n}$ with $\lambda[A] \in \Gamma$, where $\mathcal{S}^{n\times n}$ is the set of $n \times n$ symmetric matrices.

Throughout the paper we assume $A[u]$ is smooth on $\bar{M}_T$ for $u \in C^\infty(\bar{M}_T)$, $\psi \in C^\infty(T^*\bar{M} \times [0,T] \times \mathbb{R})$ (for convenience we shall write $\psi = \psi(x,t,z,p)$ for $(x,p) \in T^*\bar{M}$, $t \in [0,T]$ and $z \in \mathbb{R}$ though). Note that for fixed $(x,t) \in \bar{M}_T$ and $p \in T^*_x M$,

$$A(x,t,p) : T^*_x M \times T^*_x M \to \mathbb{R}$$

is a symmetric bilinear map. We shall use the notation

$$A^{\xi\eta}(x,\cdot,\cdot) := A(x,\cdot,\cdot)(\xi,\eta), \ \ \xi,\eta \in T^*_x M$$

and, for a function $v \in C^{2,1}_{x,t}(M_T)$, $A[v] := A(x,t,\nabla v)$, $A^{\xi\eta}[v] := A^{\xi\eta}(x,t,\nabla v)$ (see [7]).

In this paper we assume that there exists an admissible function $\underline{u} \in C^2(\bar{M}_T)$ satisfying

$$f(\lambda(\nabla^2\underline{u} + A[\underline{u}])) - \underline{u}_t \geq \psi(x,t,\underline{u},\nabla\underline{u}) + \delta_0 \text{ in } M\times[0,T]. \tag{1.7}$$

for some positive constant δ_0 with $\underline{u} = \varphi$ on $\partial M \times [0,T]$ and $\underline{u} \leq \varphi$ in $M \times \{0\}$.

We shall prove the following Theorem.

Theorem 1.1. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of (1.1). Suppose* (1.4)-(1.6) *and* (1.7) *hold. Assume that*

$$-\psi(x,t,z,p) \text{ and } A^{\xi\xi}(x,t,p) \text{ are concave in } p, \ \ \forall\, \xi \in T_x M, \tag{1.8}$$

$$\psi_z \leq 0. \tag{1.9}$$

Then

$$\max_{\bar{M}_T} |\nabla^2 u| \leq C_1\big(1 + \max_{\mathcal{P}M_T} |\nabla^2 u|\big) \tag{1.10}$$

where $C_1 > 0$ depends on $|u|_{C^1_x(\bar{M}_T)}$ and $|\underline{u}|_{C^2(\bar{M}_T)}$. Suppose that u also satisfies the boundary condition (1.2) *and, in addition, assume that*

$$\sum f_i(\lambda)\lambda_i \geq 0, \ \forall\, \lambda \in \Gamma, \tag{1.11}$$

$$f(\lambda(\nabla^2\varphi(x,0) + A[\varphi(x,0)])) - \varphi_t(x,0) = \psi[\varphi(x,0)], \quad \forall x \in \bar{M}, \tag{1.12}$$

and

$$\varphi_t(x,t) + \psi(x,t,z,p) > 0 \tag{1.13}$$

for each $(x,t) \in SM_T$, $p \in T^*_x\bar{M}$ and $z \in \mathbb{R}$. Then there exists $C_2 > 0$ depending on $|u|_{C^1_x(\bar{M}_T)}$, $|\underline{u}|_{C^2(\bar{M}_T)}$ and $|\varphi|_{C^4(\mathcal{P}M_T)}$ such that

$$\max_{\mathcal{P}M_T} |\nabla^2 u| \leq C_2. \tag{1.14}$$

Since u is admissible, we have, by (1.8),

$$\triangle u + \mathrm{tr}A_{p_k}(x,t,0)\nabla_k u + \mathrm{tr}A(x,t,0) \geq \triangle u + \mathrm{tr}A(x,t,\nabla u) > 0$$

and by the maximum principle it is easy to derive the estimate

$$\max_{\bar{M}_T} |u| + \max_{\mathcal{P}M_T} |\nabla u| \leq C. \tag{1.15}$$

Combining with the gradient estimates (Theorem 5.1-5.3), we can prove the following theorem immediately.

Theorem 1.2. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of (1.1) in M_T with $u \geq \underline{u}$ in M_T and $u = \varphi$ on $\mathcal{P}M_T$. Suppose* (1.4)-(1.6), (1.7)-(1.9), *and* (1.11)-(1.13) *hold. Then we have*

$$|u|_{C^{2,1}_{x,t}(\bar{M}_T)} \leq C, \tag{1.16}$$

where $C > 0$ depends on n, M and $|\underline{u}|_{C^2(\bar{M}_T)}$ under any of the following additional assumptions: (i) (5.1)-(5.3) *hold for $\gamma_1 < 4$, $\gamma_2 = 2$ in* (5.1)*; (ii)* (M^n, g) *has nonnegative sectional curvature and* (5.1) *hold for $\gamma_1, \gamma_2 < 2$; (iii)* (5.1), (5.16)-(5.20) *hold for $\gamma_1, \gamma_2 < 4$ in* (5.1) *and $\gamma < 2$ in* (5.18)-(5.20).

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and present a brief review of some elementary formulas. In Section 3 and Section 4, we establish the global and boundary estimates for second order derivatives respectively. The gradient estimates are derived in Section 5.

2. Preliminaries

Throughout the paper ∇ denotes the Levi-Civita connection of (M^n, g). The curvature tensor is defined by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Let $e_1, \ldots, e_n$ be local frames on M^n. We denote $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$. Define the Christoffel symbols Γ^k_{ij} by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$ and the curvature coefficients

$$R_{ijkl} = g(R(e_k, e_l)e_j, e_i), \quad R^i_{jkl} = g^{im} R_{mjkl}.$$

We shall use the notation $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k$, etc.

For a differentiable function v defined on M^n, we usually identity ∇v with the gradient of v, and use $\nabla^2 v$ to denote the Hessian of v which is locally given by $\nabla_{ij} v = \nabla_i (\nabla_j v) - \Gamma^k_{ij} \nabla_k v$. We recall that $\nabla_{ij} v = \nabla_{ji} v$ and

$$\nabla_{ijk} v - \nabla_{jik} v = R^l_{kij} \nabla_l v, \tag{2.1}$$

$$\begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v = {} & R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v + R^m_{lik} \nabla_{jm} v \\ & + R^m_{jik} \nabla_{lm} v + R^m_{jil} \nabla_{km} v + \nabla_k R^m_{jil} \nabla_m v. \end{aligned} \tag{2.2}$$

Let $u \in C^4(\bar{M}_T)$ be an admissible solution of equation (1.1). For simplicity we shall denote $U := \nabla^2 u + A(x, t, \nabla u)$ and, under a local frame $e_1, \ldots, e_n$,

$$U_{ij} \equiv U(e_i, e_j) = \nabla_{ij} u + A^{ij}(x, t, \nabla u),$$

$$\begin{aligned} \nabla_k U_{ij} \equiv \nabla U(e_i, e_j, e_k) & = \nabla_{kij} u + \nabla_k A^{ij}(x, t, \nabla u) \\ & \equiv \nabla_{kij} u + A^{ij}_{x_k}(x, t, \nabla u) + A^{ij}_{p_l}(x, t, \nabla u) \nabla_{kl} u, \end{aligned} \tag{2.3}$$

$$\begin{aligned}(U_{ij})_t &\equiv (U(e_i, e_j))_t = (\nabla_{ij}u)_t + A^{ij}_t(x,t,\nabla u) + A^{ij}_{p_l}(x,t,\nabla u)(\nabla_l u)_t \\ &\equiv \nabla_{ij}u_t + A^{ij}_t(x,t,\nabla u) + A^{ij}_{p_l}(x,t,\nabla u)\nabla_l u_t,\end{aligned} \tag{2.4}$$

where $A^{ij} = A^{e_ie_j}$ and $A^{ij}_{x_k}$ denotes the *partial* covariant derivative of A when viewed as depending on $x \in M$ only, while the meanings of A^{ij}_t and $A^{ij}_{p_l}$, etc are obvious. Similarly we can calculate $\nabla_{kl}U_{ij} = \nabla_k\nabla_l U_{ij} - \Gamma^m_{kl}\nabla_m U_{ij}$, etc.

Let F be the function defined by

$$F(h) = f(\lambda(h))$$

for a $(0,2)$ tensor h on M.

Following the literature we denote throughout this paper

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}(U)$$

under an orthonormal local frame $e_1, \ldots, e_n$. The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \ldots, f_n$ and is positive definite by assumption (1.4), while (1.5) implies that F is a concave function of U_{ij} (see [2]). Moreover, when $\{U_{ij}\}$ is diagonal so is $\{F^{ij}\}$, and the following identities hold

$$F^{ij}U_{ij} = \sum f_i\lambda_i, \quad F^{ij}U_{ik}U_{kj} = \sum f_i\lambda_i^2, \quad \lambda(U) = (\lambda_1, \ldots, \lambda_n).$$

Define the linear operator $\mathcal{L}$ locally by

$$\mathcal{L}v = F^{ij}\nabla_{ij}v + (F^{ij}A^{ij}_{p_k} - \psi_{p_k})\nabla_k v - v_t,$$

for $v \in C^{2,1}_{x,t}(M_T)$. We can prove

Theorem 2.1. *Let u be an admissible solution to* (1.1) *with $u \geq \underline{u}$ in M_T. Assume that* (1.4), (1.5), (1.8) *and* (1.9) *hold. Then there exists a constant $\theta > 0$ depending only on δ_0 and $\underline{u}$ such that*

$$\mathcal{L}(\underline{u} - u) \geq \theta(1 + \sum F^{ii}) \tag{2.5}$$

Proof. Since $\underline{u}$ is admissible satisfying (1.7), there exists a constant $\varepsilon_0 > 0$ such that $\{x \in \bar{M}_T : \lambda(\nabla^2\underline{u} + A[\underline{u}] - \varepsilon_0 g)\}$ is a compact subset of Γ and

$$f(\lambda(\nabla^2\underline{u} + A[\underline{u}] - \varepsilon_0 g)) - \underline{u}_t \geq \psi[\underline{u}] + \frac{\delta_0}{2} \text{ in } M_T.$$

Let $\theta = \min\{\frac{\delta_0}{2}, \varepsilon_0\}$. For each $(x,t) \in M_T$, we may assume $\{U_{ij}\} = \{\nabla_{ij}u + A^{ij}\}$ is diagonal at (x,t). From (1.8), (1.9) and the concavity of F, we see, at (x,t),

$$\begin{aligned}F^{ii}(\underline{U}_{ii} - \varepsilon_0 g_{ii} - U_{ii}) - (\underline{u} - u)_t &\geq \psi(x,t,\underline{u},\nabla\underline{u}) - \psi(x,t,u,\nabla u) + \frac{\delta_0}{2} \\ &\geq \psi(x,t,u,\nabla\underline{u}) - \psi(x,t,u,\nabla u) + \frac{\delta_0}{2} \\ &\geq \psi_{p_k}\nabla_k(\underline{u} - u) + \frac{\delta_0}{2}.\end{aligned} \tag{2.6}$$

By (1.8) again, we have

$$\begin{aligned} F^{ii}(\underline{U}_{ii} - U_{ii}) &= F^{ii}\nabla_{ii}(\underline{u} - u) + F^{ii}(A^{ii}(x,t,\nabla\underline{u}) - A^{ii}(x,t,\nabla u)) \\ &\geq F^{ii}\nabla_{ii}(\underline{u} - u) + F^{ii}A^{ii}_{p_k}\nabla_k(\underline{u} - u). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\mathcal{L}(\underline{u} - u) \geq \varepsilon_0 \sum F^{ii} + \frac{\delta_0}{2} \geq \theta(1 + \sum F^{ii})$$

□

3. Global estimates for second derivatives

In this section, we prove (1.10) in Theorem 1.1 for which we set

$$W = \max_{(x,t)\in\bar{M}_T} \max_{\xi\in T_xM, |\xi|=1} (\nabla_{\xi\xi}u + A^{\xi\xi}(x,u,\nabla u)e^{\phi},$$

as in [7], where ϕ is a function to be determined. It suffices to estimate W. We may assume W is achieved at $(x_0,t_0) \in \bar{M}_T - \mathcal{P}M_T$. Choose a smooth orthonormal local frame $e_1,\ldots,e_n$ about x_0 such that $\nabla_i e_j = 0$, and U is diagonal at (x_0,t_0). We assume $U_{11}(x_0,t_0) \geq \ldots \geq U_{nn}(x_0,t_0)$. We have $W = U_{11}(x_0,t_0)e^{\phi(x_0,t_0)}$.

At the point (x_0,t_0) where the function $\log U_{11} + \phi$ attains its maximum, we have

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i\phi = 0 \text{ for each } i = 1,\ldots,n, \tag{3.1}$$

$$\frac{(U_{11})_t}{U_{11}} + \phi_t \geq 0, \tag{3.2}$$

and

$$0 \geq \sum_i F^{ii}\{\frac{\nabla_{ii}U_{11}}{U_{11}} - \frac{(\nabla_i U_{11})^2}{U_{11}^2} + \nabla_{ii}\phi\}. \tag{3.3}$$

Differentiating equation (1.1) twice, we find

$$F^{ii}\nabla_k U_{ii} - \nabla_k u_t = \psi_{x_k} + \psi_u\nabla_k u + \psi_{p_j}\nabla_{kj}u, \text{ for all } k, \tag{3.4}$$

and

$$\begin{aligned} F^{ii}\nabla_{11}U_{ii} + &F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - \nabla_{11}u_t \\ &\geq \psi_{p_j}\nabla_{11j}u + \psi_{p_l p_k}\nabla_{1k}u\nabla_{1l}u - CU_{11} \\ &\geq \psi_{p_j}\nabla_j U_{11} + \psi_{p_1p_1}U_{11}^2 - CU_{11} \\ &= -U_{11}\psi_{p_j}\nabla_j\phi + \psi_{p_1p_1}U_{11}^2 - CU_{11}. \end{aligned} \tag{3.5}$$

Next, by (3.1) and (3.4),

$$
\begin{aligned}
F^{ii}(\nabla_{ii}A^{11} - \nabla_{11}A^{ii}) \geq\, & F^{ii}(A^{11}_{p_j}\nabla_{iij}u - A^{ii}_{p_j}\nabla_{11j}u) \\
& + F^{ii}(A^{11}_{p_ip_i}U^2_{ii} - A^{ii}_{p_1p_1}U^2_{11}) - CU_{11}\sum F^{ii} \\
\geq\, & U_{11}F^{ii}A^{ii}_{p_j}\nabla_j\phi + A^{11}_{p_j}\nabla_j u_t - CU_{11}\sum F^{ii} - CU_{11} \\
& - C\sum_{i\geq 2} F^{ii}U^2_{ii} - U^2_{11}\sum_{i\geq 2} F^{ii}A^{ii}_{p_1p_1}.
\end{aligned} \tag{3.6}
$$

Note that

$$
\nabla_{ii}U_{11} \geq \nabla_{11}U_{ii} + \nabla_{ii}A^{11} - \nabla_{11}A^{ii} - CU_{11}. \tag{3.7}
$$

Thus, by (3.5), (3.6) and (3.2), we have, at (x_0, t_0),

$$
\begin{aligned}
F^{ii}\nabla_{ii}U_{11} \geq\, & F^{ii}\nabla_{11}U_{ii} - CU_{11}(1 + \sum F^{ii}) + A^{11}_{p_j}\nabla_j u_t \\
& - C\sum_{i\geq 2} F^{ii}U^2_{ii} - U^2_{11}\sum_{i\geq 2} F^{ii}A^{ii}_{p_1p_1} + U_{11}F^{ii}A^{ii}_{p_j}\nabla_j\phi \\
\geq\, & U_{11}\mathcal{L}\phi - U_{11}F^{ii}\nabla_{ii}\phi - F^{ij,kl}\nabla_1U_{ij}\nabla_1U_{kl} + \psi_{p_1p_1}U^2_{11} \\
& - CU_{11}(1 + \sum F^{ii}) - CF^{ii}U^2_{ii} - U^2_{11}\sum_{i\geq 2} F^{ii}A^{ii}_{p_1p_1}.
\end{aligned} \tag{3.8}
$$

It follows that, by (3.3),

$$
\mathcal{L}\phi \leq U_{11}\sum_{i\geq 2} F^{ii}A^{ii}_{p_1p_1} - \psi_{p_1p_1}U_{11} + C(1 + \sum F^{ii}) + \frac{C}{U_{11}}F^{ii}U^2_{ii} + E, \tag{3.9}
$$

where

$$
E = \frac{1}{U^2_{11}}F^{ii}(\nabla_iU_{11})^2 + \frac{1}{U_{11}}F^{ij,kl}\nabla_1U_{ij}\nabla_1U_{kl}.
$$

Let

$$
\phi = \frac{\delta|\nabla u|^2}{2} + b\eta,
$$

where b, δ are undetermined constants, $0 < \delta < 1 \leq b$, and η is a C^2 function which may depend on u but not on its derivatives. We calculate, at (x_0, t_0),

$$
\nabla_i\phi = \delta\nabla_j u\nabla_{ij}u + b\nabla_i\eta = \delta\nabla_i uU_{ii} - \delta\nabla_j uA^{ij} + b\nabla_i\eta \tag{3.10}
$$

$$
\phi_t = \delta\nabla_j u(\nabla_j u)_t + b\eta_t \tag{3.11}
$$

$$
\nabla_{ii}\phi \geq \frac{\delta}{2}U^2_{ii} - C\delta + \delta\nabla_j u\nabla_{iij}u + b\nabla_{ii}\eta. \tag{3.12}
$$

From (2.1) and (3.4), we derive

$$
\begin{aligned}
F^{ii}\nabla_j u \nabla_{iij} u &\geq F^{ii}\nabla_j u(\nabla_j U_{ii} - \nabla_j A^{ii}) - C|\nabla u|^2 \sum F^{ii} \\
&\geq (\psi_{p_l} - F^{ii}A^{ii}_{p_l})\nabla_j u \nabla_{jl} u + \nabla_j u \nabla_j (u_t) - C(1 + \sum F^{ii}).
\end{aligned} \tag{3.13}
$$

Therefore,

$$
\mathcal{L}\phi \geq b\mathcal{L}\eta + \frac{\delta}{2}F^{ii}U_{ii}^2 - C\sum F^{ii} - C. \tag{3.14}
$$

Let $\eta = \underline{u} - u$. We get from (3.10) that

$$
(\nabla_i \phi)^2 \leq C\delta^2(1 + U_{ii}^2) + 2b^2(\nabla_i \eta)^2 \leq C\delta^2 U_{ii}^2 + Cb^2. \tag{3.15}
$$

For fixed $0 < s \leq 1/3$ let

$$
J = \{i : U_{ii} \leq -sU_{11}\}, \quad K = \{i : U_{ii} > -sU_{11}\}.
$$

Using a result of Andrews [1] and Gerhardt [3] as in [5] and [7] (see [13] also), we have

$$
E \leq Cb^2 \sum_{i\in J} F^{ii} + C\delta^2 \sum F^{ii}U_{ii}^2 + C\sum F^{ii} + C(\delta^2 U_{11}^2 + b^2)F^{11}. \tag{3.16}
$$

Therefore, by (3.9), (3.14) and (3.16), we have

$$
\begin{aligned}
b\mathcal{L}\eta &\leq \Big(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}\Big)F^{ii}U_{ii}^2 + Cb^2\sum_{i\in J}F^{ii} + C\sum F^{ii} \\
&\quad + C(\delta^2 U_{11}^2 + b^2)F^{11} + C \\
&\leq \Big(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}\Big)F^{ii}U_{ii}^2 + Cb^2\sum_{i\in J}F^{ii} + C\sum F^{ii} \\
&\quad + Cb^2F^{11} + C.
\end{aligned} \tag{3.17}
$$

Choose δ sufficiently small such that $C\delta^2 - \frac{\delta}{2}$ is negative and let

$$
c_1 := -\frac{1}{2}\Big(C\delta^2 - \frac{\delta}{2}\Big) > 0.
$$

We may assume

$$
C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \leq -c_1
$$

for otherwise we have $U_{11} \leq \frac{C}{c_1}$ and we are done. Thus, by (2.5), choosing b sufficiently large, we derive from (3.17) that

$$
c_1 F^{ii}U_{ii}^2 - Cb^2F^{11} - Cb^2\sum_{i\in J}F^{ii} \leq 0.
$$

Then we can get a bound $U_{11}(x_0, t_0) \leq C$ since $|U_{ii}| \geq sU_{11}$ for $i \in J$. The proof of (1.10) is completed.

4. Boundary estimates for second derivatives

In this section, we consider the estimates of second order derivatives on parabolic boundary $\mathcal{P}M_T$. We may assume $\varphi \in C^4(\bar{M}_T)$.

Fix a point $(x_0, t_0) \in SM_T$. We shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around x_0 such that when restricted to ∂M, e_n is normal to ∂M. Since $u - \underline{u} = 0$ on SM_T we have

$$\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u})\mathit{\Pi}(e_\alpha, e_\beta), \ \ \forall\, 1 \leq \alpha, \beta < n \ \text{ on } SM_T, \tag{4.1}$$

where $\mathit{\Pi}$ denotes the second fundamental form of ∂M. Therefore,

$$|\nabla_{\alpha\beta} u| \leq C, \ \ \forall\, 1 \leq \alpha, \beta < n \ \text{ on } \ SM_T. \tag{4.2}$$

Let $\rho(x)$ denote the distance from $x \in M$ to x_0,

$$\rho(x) \equiv \mathrm{dist}_{M^n}(x, x_0),$$

and set

$$M_\delta = \{X = (x,t) \in M \times (0,T] : \rho(x) < \delta, t \leq t_0 + \delta\}.$$

For the mixed tangential-normal and pure normal second derivatives at (x_0, t_0), we shall use the following barrier function as in [5],

$$\mathit{\Psi} = A_1 v + A_2 \rho^2 - A_3 \sum_{l<n} |\nabla_l (u - \varphi)|^2 \tag{4.3}$$

where $v = u - \underline{u}$. By differentiating the equation (1.1) and straightforward calculation, we obtain

$$\mathcal{L}(\nabla_k(u - \varphi)) \leq C\Big(1 + \sum f_i |\lambda_i| + \sum f_i\Big), \ \ \forall\, 1 \leq k \leq n. \tag{4.4}$$

Similar to [5] (see [7] also), using Proposition 2.19 and Corollary 2.21 of [5] and Theorem 2.1, we can prove that there exist uniform positive constants δ sufficiently small, and A_1, A_2, A_3 sufficiently large such that

$$\mathcal{L}(\mathit{\Psi} \pm \nabla_\alpha(u - \varphi)) \leq 0 \text{ in } M_\delta \tag{4.5}$$

and $\mathit{\Psi} \pm \nabla_\alpha(u - \varphi) \geq 0$ on $\mathcal{P}M_\delta$. Thus, by the maximum principle, we see $\mathit{\Psi} \pm \nabla_\alpha(u - \varphi) \geq 0$ in M_δ. Then we get

$$|\nabla_{n\alpha} u(x_0, t_0)| \leq \nabla_n \mathit{\Psi}(x_0, t_0) \leq C, \ \ \forall\, \alpha < n. \tag{4.6}$$

It remains to derive

$$\nabla_{nn} u(x_0, t_0) \leq C \tag{4.7}$$

since $\triangle u \geq -C$. We shall use an idea of Trudinger [12] as [5] and [7] to prove that there exist uniform positive constants c_0, R_0 such that for all $R > R_0$, $(\lambda'[U], R) \in \Gamma$ and

$$f(\lambda'[U], R) - u_t \geq \psi[u] + c_0 \text{ on } \overline{SM_T} \tag{4.8}$$

which implies (4.7) by Lemma 1.2 in [2], where $\lambda'[U] = (\lambda'_1, \ldots, \lambda'_{n-1})$ denote the eigenvalues of the $(n-1)\times(n-1)$ matrix $\{U_{\alpha\beta}\}_{1\le\alpha,\beta\le(n-1)}$ and $\psi[u] = \psi(\cdot,\cdot,u,\nabla u)$. For $R > 0$ and a symmetric $(n-1)^2$ matrix $\{r_{\alpha\beta}\}$ with $(\lambda'(\{r_{\alpha\beta}\}), R) \in \Gamma$, define

$$G[r_{\alpha\beta}] \equiv f(\lambda'[\{r_{\alpha\beta}\}], R)$$

and consider

$$m_R \equiv \min_{(x,t)\in\overline{SM_T}} G[U_{\alpha\beta}(x,t)] - u_t(x,t) - \psi[u].$$

Note that G is concave and m_R is increasing in R by (1.4), and that

$$\begin{aligned} c_R &\equiv \inf_{\overline{SM_T}} (G[\underline{U}_{\alpha\beta}] - \underline{u}_t - \psi[\underline{u}]) \\ &\ge \inf_{\overline{SM_T}} (G[\underline{U}_{\alpha\beta}] - F[\underline{U}_{ij}]) > 0 \end{aligned}$$

when R is sufficiently large.

We wish to show $m_R > 0$ for R sufficiently large. Without loss of generality we assume $m_R < c_R/2$ (otherwise we are done) and suppose m_R is achieved at a point $(x_0,t_0) \in \overline{SM_T}$. Choose local orthonormal frames around x_0 as before and assume $\nabla_{nn}u(x_0,t_0) \ge \nabla_{nn}\underline{u}(x_0,t_0)$. Let $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n\rangle$ and

$$G_0^{\alpha\beta} = \frac{\partial G}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0,t_0)].$$

Note that $\sigma_{\alpha\beta} = \mathit{\Pi}(e_\alpha, e_\beta)$ on ∂M and that

$$G_0^{\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0,t_0)) \ge G[r_{\alpha\beta}] - G[U_{\alpha\beta}(x_0,t_0)] \tag{4.9}$$

for any symmetric matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$ by the concavity of G.

In particular, since $u_t = \underline{u}_t = \varphi_t$ on $\overline{SM_T}$, we have

$$\begin{aligned} G_0^{\alpha\beta}U_{\alpha\beta} - \psi[u] - \varphi_t\, - G_0^{\alpha\beta}U_{\alpha\beta}(x_0,t_0) + \psi[u](x_0,t_0) + u_t(x_0,t_0) \\ \ge G[U_{\alpha\beta}] - \psi[u] - u_t - m_R \ge 0 \end{aligned} \tag{4.10}$$

on $\overline{SM_T}$.

From (4.1) we see that

$$U_{\alpha\beta} = \underline{U}_{\alpha\beta} - \nabla_n(u-\underline{u})\sigma_{\alpha\beta} + A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}] \text{ on } \overline{SM_T}. \tag{4.11}$$

Note that at (x_0, t_0), we have

$$
\begin{aligned}
\nabla_n(u-\underline{u})G_0^{\alpha\beta}\sigma_{\alpha\beta} &= G_0^{\alpha\beta}(\underline{U}_{\alpha\beta}-U_{\alpha\beta}) + G_0^{\alpha\beta}(A^{\alpha\beta}[u]-A^{\alpha\beta}[\underline{u}]) \\
&\geq G[\underline{U}_{\alpha\beta}] - G[U_{\alpha\beta}] + G_0^{\alpha\beta}(A^{\alpha\beta}[u]-A^{\alpha\beta}[\underline{u}]) \\
&= G[\underline{U}_{\alpha\beta}] - \psi[u] - u_t - m_R + G_0^{\alpha\beta}(A^{\alpha\beta}[u]-A^{\alpha\beta}[\underline{u}]) \\
&\geq c_R - m_R + \psi[\underline{u}] + \underline{u}_t - \psi[u] - u_t \\
&\quad + G_0^{\alpha\beta}(A^{\alpha\beta}[u]-A^{\alpha\beta}[\underline{u}]) \\
&\geq \frac{c_R}{2} + H[u] - H[\underline{u}]
\end{aligned}
\tag{4.12}
$$

where $H[u] = G_0^{\alpha\beta}A^{\alpha\beta}[u] - \psi[u]$.

Define

$$\Phi = -\eta\nabla_n(u-\underline{u}) + H[u] - \varphi_t + Q$$

where $\eta = G_0^{\alpha\beta}\sigma_{\alpha\beta}$ and

$$Q \equiv G_0^{\alpha\beta}\nabla_{\alpha\beta}\underline{u} - G_0^{\alpha\beta}U_{\alpha\beta}(x_0,t_0) + \psi[u](x_0,t_0) + u_t(x_0,t_0).$$

By virtue of (4.10) and (4.11) we see that $\Phi \geq 0$ on $\overline{SM_T}$ and $\Phi(x_0,t_0)=0$.

Next, by (4.4) and (1.8),

$$
\begin{aligned}
\mathcal{L}H \leq{}& H_z[u]\mathcal{L}u + H_{p_k}[u]\mathcal{L}\nabla_k u + F^{ij}H_{p_kp_l}[u]\nabla_{ki}u\nabla_{lj}u \\
&+ C(\sum F^{ii} + \sum f_i|\lambda_i| + 1) \\
\leq{}& C(\sum F^{ii} + \sum f_i|\lambda_i| + 1) + H_z[u]\mathcal{L}u.
\end{aligned}
$$

Since $H_z[u] \geq 0$, by Theorem 2.1, we have

$$\mathcal{L}u = \mathcal{L}(u-\underline{u}) + \mathcal{L}\underline{u} \leq C(1+\sum F^{ii}).$$

It follows that

$$\mathcal{L}H \leq C(\sum F^{ii} + \sum f_i|\lambda_i| + 1).$$

Therefore,

$$\mathcal{L}\Phi \leq C(\sum F^{ii} + \sum f_i|\lambda_i| + 1). \tag{4.13}$$

By the compatibility condition(1.12), we find that

$$c_R' \equiv \inf_{x\in\bar{M}} G(\nabla_{\alpha\beta}\varphi + A[\varphi])(x,0) - \psi[\varphi](x,0) - \varphi_t(x,0) > 0$$

when R is sufficiently large. We may assume $m_R < \frac{c'_R}{2}$ (otherwise we are done). For $x \in \bar{M}$, by the concavity of G again, we have

$$\begin{aligned}
\Phi(x,0) =& G_0^{\alpha\beta}(U_{\alpha\beta}(x,0) - U_{\alpha\beta}(x_0,t_0)) \\
&- \psi[u](x,0) - \varphi_t(x,0) + \psi[u](x_0,t_0) + u_t(x_0,t_0) \\
=& G_0^{\alpha\beta}(\nabla_{\alpha\beta}\varphi + A[\varphi](x,0) - U_{\alpha\beta}(x_0,t_0)) \\
&- \varphi_t(x,0) + u_t(x_0,t_0) + \psi[u](x_0,t_0) - \psi[\varphi](x,0) \\
\geq& G(\nabla_{\alpha\beta}\varphi + A[\varphi])(x,0) - G(U_{\alpha\beta}(x_0,t_0)) \\
&- \varphi_t(x,0) + u_t(x_0,t_0) + \psi[u](x_0,t_0) - \psi[\varphi](x,0) \\
\geq& c'_R - m_R > \frac{c'_R}{2}.
\end{aligned}$$

It means that $\Phi > 0$ on BM_T. Thus, we get $\Phi \geq 0$ on $\mathcal{P}M_\delta$.

Consider the function Ψ defined in (4.3) as before. Similarly, there exist another group of constants $A_1 \gg A_2 \gg A_3 \gg 1$ such that

$$\begin{cases} \mathcal{L}(\Psi + \Phi) \leq 0 \quad \text{in } M_\delta, \\ \Psi + \Phi \geq 0 \quad \text{on } \mathcal{P}M_\delta. \end{cases} \tag{4.14}$$

By the maximum principle we find $\Psi + \Phi \geq 0$ in M_δ. It follows that $\nabla_n\Phi(x_0,t_0) \geq -\nabla_n\Psi(x_0,t_0) \geq -C$.

Following [7], we write $u^s = su + (1-s)\underline{u}$ and

$$H[u^s] = G_0^{\alpha\beta}A^{\alpha\beta}[u^s] - \psi[u^s].$$

We have

$$\begin{aligned}
H[u] - H[\underline{u}] =& \int_0^1 \frac{dH[u^s]}{dt} ds \\
=& (u - \underline{u}) \int_0^1 H_z[u^s] ds + \sum \nabla_k(u - \underline{u}) \int_0^1 H_{p_k}[u^s] ds.
\end{aligned}$$

Therefore, at (x_0,t_0),

$$H[u] - H[\underline{u}] = \nabla_n(u - \underline{u}) \int_0^1 H_{p_n}[u^s] ds \tag{4.15}$$

and

$$
\begin{aligned}
\nabla_n H[u] = {} & \nabla_n H[\underline{u}] + \sum \nabla_{kn}(u-\underline{u}) \int_0^1 H_{p_k}[u^s]ds \\
& + \nabla_n(u-\underline{u}) \int_0^1 (H_z[u^s] + H_{x_n p_n}[u^s] + H_{zp_n}[u^s]\nabla_n u^s)ds \\
& + \nabla_n(u-\underline{u}) \sum \int_0^1 H_{p_n p_l}[u^s]\nabla_{ln} u^s ds \\
\le {} & \nabla_{nn}(u-\underline{u}) \int_0^1 (H_{p_n}[u^s] + sH_{p_n p_n}[u^s]\nabla_n(u-\underline{u}))ds + C \\
\le {} & \nabla_{nn}(u-\underline{u}) \int_0^1 H_{p_n}[u^s]ds + C
\end{aligned} \tag{4.16}
$$

since $H_{p_n p_n} \le 0$, $\nabla_{nn}(u-\underline{u}) \ge 0$ and $\nabla_n(u-\underline{u}) \ge 0$. It follows that

$$
\begin{aligned}
\nabla_n \Phi(x_0,t_0) \le {} & -\eta(x_0,t_0)\nabla_{nn}(x_0,t_0) + \nabla_n H[u](x_0,t_0) + C \\
\le {} & \Big(-\eta(x_0,t_0) + \int_0^1 H_{p_n}[u^s](x_0,t_0)ds \Big) \nabla_{nn} u(x_0,t_0) + C.
\end{aligned} \tag{4.17}
$$

By (4.12) and (4.15),

$$
\eta(x_0,t_0) - \int_0^1 H_{p_n}[u^s](x_0,t_0)ds \ge \frac{c_R}{2\nabla_n(u-\underline{u})(x_0,t_0)} \ge \epsilon_1 c_R > 0 \tag{4.18}
$$

for some uniform $\epsilon_1 > 0$ independent of R. This gives

$$
\nabla_{nn} u(x_0,t_0) \le \frac{C}{\epsilon_1 c_R}. \tag{4.19}
$$

So we have an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0,t_0)\}$. Now by (1.13), there exists a constant $\nu_0 > 0$ such that

$$
\inf_{(x,t)\in\overline{SM_T}} \varphi_t(x,t) + \psi(x,t,u,\nabla u) \ge \nu_0.
$$

It follows that $\lambda[\{U_{ij}(x_0,t_0)\}]$ is contained in a compact subset of Γ by (1.6), and therefore

$$
m_R = G[U_{\alpha\beta}(x_0,t_0)] - u_t(x_0,t_0) - \psi[u](x_0,t_0) > 0
$$

when R is sufficiently large. Then (4.8) is valid and the proof of (1.14) is completed.

5. Gradient estimates

In this section we establish the gradient estimates to prove Theorem 5.1-5.3 below. Throughout the section, we assume (1.4)-(1.5), (1.8) and the following growth conditions hold

$$
\begin{cases}
p \cdot \nabla_x A^{\xi\xi}(x,t,z,p) \le \bar{\psi}_1(x,t,z)|\xi|^2(1+|p|^{\gamma_1}) \\
p \cdot \nabla_x \psi(x,t,z,p) + |p|^2\psi_z(x,t,z,p) \ge -\bar{\psi}_2(x,t,z)(1+|p|^{\gamma_2})
\end{cases} \tag{5.1}
$$

for some functions $\bar{\psi}_1, \bar{\psi}_2 \geq 0$ and constants $\gamma_1, \gamma_2 > 0$.

Since the proofs of Theorem 5.1-5.3 are similar to those of Theorem 6.1-6.3 in [7], we only provide a sketch here. For more details we refer the reader to [7] where the elliptic Hessian equations are treated.

Theorem 5.1. *Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.1). Assume, in addition, that*

$$\lim_{\sigma\to\infty} f(\sigma\mathbf{1}) = +\infty \tag{5.2}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and there exists a constant $c_0 > 0$ such that

$$A^{\xi\xi}_{p_k p_l}(x, t, p)\eta_k\eta_l \leq -c_0|\xi|^2|\eta|^2 + c_0|g(\xi, \eta)|^2, \ \forall\, \xi, \eta \in T_x M. \tag{5.3}$$

Suppose that $\gamma_1 < 4$, $\gamma_2 = 2$ in (5.1), *and that there is an admissible function $\underline{u} \in C^2(\bar{M}_T)$. Then*

$$\max_{\bar{M}_T} |\nabla u| \leq C_3\big(1 + \max_{\mathcal{P}M_T} |\nabla u|\big) \tag{5.4}$$

where C_3 is a positive constant depending on $|u|_{C^0(\bar{M}_T)}$ and $|\underline{u}|_{C^1_x(\bar{M}_T)}$.

Proof. Let $w = |\nabla u|$ and ϕ a positive function to be determined. Suppose the function $w\phi^{-a}$ achieves a positive maximum at an interior point $(x_0, t_0) \in M_T - \mathcal{P}M_T$ where $a < 1$ is a positive constant. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ at x_0 and $\{U_{ij}(x_0, t_0)\}$ is diagonal.

The function $\log w - a \log \phi$ attains its maximum at (x_0, t_0) where for $i = 1, \ldots, n$,

$$\frac{\nabla_i w}{w} - \frac{a\nabla_i \phi}{\phi} = 0, \tag{5.5}$$

$$\frac{w_t}{w} - \frac{a\phi_t}{\phi} \geq 0 \tag{5.6}$$

and

$$\frac{\nabla_{ii} w}{w} + \frac{(a - a^2)|\nabla_i \phi|^2}{\phi^2} - \frac{a\nabla_{ii}\phi}{\phi} \leq 0. \tag{5.7}$$

Note that

$$w\nabla_i w = \nabla_l u \nabla_{il} u, \ \ w w_t = \nabla_l u (\nabla_l u)_t.$$

By (2.1), (5.5) and (3.4),

$$\begin{aligned} w\nabla_{ii} w &= \nabla_l u \nabla_{iil} u + \nabla_{il} u \nabla_{il} u - \nabla_i w \nabla_i w \\ &= (\nabla_{lii} u + R^k_{iil} \nabla_k u)\nabla_l u + \Big(\delta_{kl} - \frac{\nabla_k u \nabla_l u}{w^2}\Big)\nabla_{ik} u \nabla_{il} u \\ &\geq (\nabla_l U_{ii} - A^{ii}_{p_k} \nabla_{lk} u - A^{ii}_{x_l})\nabla_l u - C|\nabla u|^2 \\ &= \nabla_l u \nabla_l U_{ii} - \frac{a w^2}{\phi} A^{ii}_{p_k} \nabla_k \phi - \nabla_l u A^{ii}_{x_l} - C w^2. \end{aligned} \tag{5.8}$$

By (3.4), (5.5) and (5.6),

$$
\begin{aligned}
F^{ii}\nabla_l u\nabla_l U_{ii} &= \nabla_l u\psi_{x_l} + \psi_u|\nabla u|^2 + \psi_{p_k}\nabla_l u\nabla_{lk}u + \nabla_l u\nabla_l u_t \\
&\geq \nabla_l u\psi_{x_l} + \psi_u|\nabla u|^2 + \frac{aw^2}{\phi}\psi_{p_k}\nabla_k\phi + \frac{aw^2}{\phi}\phi_t.
\end{aligned} \tag{5.9}
$$

Let $\phi = (u - \underline{u}) + b > 0$, where $b = 1 + \sup_{M_T}(\underline{u} - u)$.

By (5.3) we have

$$
\begin{aligned}
-A^{ii}_{p_k}\nabla_k\phi &= A^{ii}_{p_k}(x,t,\nabla u)\nabla_k(\underline{u} - u) \\
&\geq A^{ii}(x,t,\nabla\underline{u}) - A^{ii}(x,t,\nabla u) + \frac{c_0}{2}(|\nabla\phi|^2 - |\nabla_i\phi|^2).
\end{aligned} \tag{5.10}
$$

We may assume that c_0 is sufficiently small and that

$$
\frac{2a - 2a^2 - c_0 a\phi}{2\phi^2} > 0
$$

by choosing a sufficiently small.

Thus, by (5.7), (5.8), (5.9) and (5.10), we find

$$
\begin{aligned}
0 &\geq \frac{a}{\phi}F^{ii}(\underline{U}_{ii} - U_{ii}) + \frac{ac_0|\nabla\phi|^2}{2\phi}\sum F^{ii} + \frac{2a - 2a^2 - c_0 a\phi}{2\phi^2}F^{ii}|\nabla_i\phi|^2 \\
&\quad - \frac{1}{w^2}F^{ii}A^{ii}_{x_l}\nabla_l u + \frac{1}{w^2}\psi_{x_l}\nabla_l u + \psi_u + \frac{a}{\phi}\psi_{p_k}\nabla_k\phi + \frac{a}{\phi}\phi_t - C\sum F^{ii} \\
&\geq \frac{a}{\phi}F^{ii}(\underline{U}_{ii} - U_{ii}) + \frac{ac_0|\nabla\phi|^2}{2\phi}\sum F^{ii} - C\sum F^{ii} \\
&\quad + \frac{a}{\phi}(\psi(x,t,u,\nabla u) - \psi(x,t,u,\nabla\underline{u})) \\
&\quad - \frac{1}{w^2}F^{ii}A^{ii}_{x_l}\nabla_l u + \frac{1}{w^2}\psi_{x_l}\nabla_l u + \psi_u + \frac{a}{\phi}(u - \underline{u})_t
\end{aligned} \tag{5.11}
$$

Choose $B > 0$ sufficiently large such that (see [7])

$$
F(2Bg + \underline{U}) \geq F(Bg) \ \text{ in } \bar{M}_T.
$$

Therefore, by the concavity of F,

$$
\begin{aligned}
F^{ii}(\underline{U}_{ii} - U_{ii}) &\geq F(2Bg + \underline{U}) - F(U) - 2B\sum F^{ii} \\
&\geq F(Bg) - 2B\sum F^{ii} - \psi(x,t,u,\nabla u) - u_t.
\end{aligned} \tag{5.12}
$$

It follows from (5.1), (5.2), (5.11) and (5.12) that

$$\begin{aligned}0 \geq& \frac{a}{\phi}F(Bg) - C - (C+2B)\sum F^{ii} + \frac{ac_0|\nabla\phi|^2}{2\phi}\sum F^{ii}\\ &- \frac{1}{w^2}F^{ii}A^{ii}_{x_l}\nabla_l u + \frac{1}{w^2}\psi_{x_l}\nabla_l u + \psi_u\\ \geq& \Big(\frac{ac_0|\nabla\phi|^2}{2\phi} - 3B - C|\nabla u|^{\gamma_1-2}\Big)\sum F^{ii}\end{aligned} \tag{5.13}$$

provided B is chosen sufficiently large. Thus, we get a bound $|\nabla u(x_0,t_0)| \leq C$ and so the proof of Theorem 5.1 is completed. □

Theorem 5.2. *Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.1) with $u \geq \underline{u}$ in M_T. Assume, in addition, that (1.7), (1.9) and (5.1) hold for $\gamma_1, \gamma_2 < 2$ in (5.1) and that (M^n, g) has nonnegative sectional curvature. Then (5.4) holds.*

Proof. Since (M^n, g) has nonnegative sectional curvature, in orthonormal local frame,

$$R^k_{iil}\nabla_k u\nabla_l u \geq 0.$$

In the proof of Theorem 5.1, similar to (5.8), we have

$$w\nabla_{ii}w \geq \nabla_l u\nabla_l U_{ii} - \frac{aw^2}{\phi}A^{ii}_{p_k}\nabla_k\phi - \nabla_l u A^{ii}_{x_l}. \tag{5.14}$$

It follows from (2.5), (5.1), (5.7), (5.9) and (5.14) that

$$\begin{aligned}0 \geq& \frac{a}{\phi}\mathcal{L}(\underline{u} - u) + \frac{1}{w^2}\nabla_l u\psi_{x_l} + \psi_u - \frac{\nabla_l u}{w^2}F^{ii}A^{ii}_{x_l} + \frac{a-a^2}{\phi^2}F^{ii}|\nabla_i\phi|^2\\ \geq& \frac{a}{\phi}\theta(1+\sum F^{ii}) - C|\nabla u|^{\gamma_1-2}\sum F^{ii} - C|\nabla u|^{\gamma_2-2} + \frac{a-a^2}{\phi^2}F^{ii}|\nabla_i\phi|^2\end{aligned} \tag{5.15}$$

provided $|\nabla u|$ is sufficiently large. Choosing a sufficiently small, we can obtain a bound $|\nabla u(x_0,t_0)| \leq C$ and (5.4) holds. □

Theorem 5.3. *Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.1) in M_T. Assume, in addition, that (5.1) hold for $\gamma_1, \gamma_2 < 4$,*

$$f \text{ is homogeneous of degree one,} \tag{5.16}$$

$$f_j(\lambda) \geq \nu_1\Big(1 + \sum f_i(\lambda)\Big) \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0, \tag{5.17}$$

where ν_1 is a uniform positive constant and there exist a continuous function $\bar{\psi} \geq 0$ and a positive constant $\gamma < 2$ such that when $|p|$ is sufficiently large,

$$p\cdot D_p\psi(x,t,z,p),\ -p\cdot D_pA^{\xi\xi}(x,t,z,p)/|\xi|^2 \leq \bar{\psi}(x,t,z)(1+|p|^\gamma), \tag{5.18}$$

$$-\psi(x,t,z,p) \leq \bar{\psi}(x,t,z)(1+|p|^\gamma), \tag{5.19}$$

$$|A^{\xi\eta}(x,t,z,p)| \leq \bar{\psi}(x,t,z)|\xi||\eta|(1+|p|^\gamma),\ \ \forall\, \xi,\eta \in T_x\bar{M}; \xi \perp \eta. \tag{5.20}$$

Then (5.4) *holds.*

Proof. In the proof of Theorem 5.1, we take $\phi = -u + \sup_{M_T} u + 1$. By the concavity of A^{ii} with respect to p,

$$A^{ii} = A^{ii}(x,t,\nabla u) \leq A^{ii}(x,t,0) + A^{ii}_{p_k}(x,t,0)\nabla_k u \tag{5.21}$$

Thus, from (5.16), (5.19) and (5.21), we find

$$\begin{aligned} -F^{ii}\nabla_{ii}\phi = F^{ii}\nabla_{ii}u = F^{ii}U_{ii} - F^{ii}A^{ii} &= u_t + \psi - F^{ii}A^{ii} \\ &\geq u_t + \psi - C(1+|\nabla u|)\sum F^{ii} \\ &\geq u_t - C(1+|\nabla u|)\sum F^{ii} - C|\nabla u|^{\gamma}. \end{aligned} \tag{5.22}$$

By virtue of (5.7), (5.8), (5.9), (5.1), (5.18) and (5.22), we see that for $a < 1$,

$$\begin{aligned} 0 \geq &\frac{(a-a^2)}{\phi^2}F^{ii}|\nabla_i u|^2 + \frac{\nabla_l u \psi_{x_l}}{w^2} + \psi_u - \frac{a}{\phi}\psi_{p_k}\nabla_k u - \frac{a}{\phi}u_t \\ &+ \frac{a}{\phi}F^{ii}A^{ii}_{p_k}\nabla_k u - F^{ii}\frac{\nabla_l u A^{ii}_{x_l}}{w^2} + \frac{a}{\phi}u_t \\ &- C|\nabla u|^{\gamma} - C(1+|\nabla u|)\sum F^{ii} \\ \geq &c_1 F^{ii}|\nabla_i u|^2 - C(|\nabla u|^{\gamma_2 - 2} + |\nabla u|^{\gamma}) \\ &- C(1 + |\nabla u| + |\nabla u|^{\gamma_1 - 2} + |\nabla u|^{\gamma})\sum F^{ii} \end{aligned} \tag{5.23}$$

provided $|\nabla u|$ is sufficiently large.

Without loss of generality we assume $\nabla_1 u(x_0,t_0) \geq \frac{1}{n}|\nabla u(x_0,t_0)| > 0$. Recall that $U_{ij}(x_0,t_0)$ is diagonal. By (5.5), (5.21) and (5.20), we have

$$\begin{aligned} U_{11} = &-\frac{a}{\phi}|\nabla u|^2 + A^{11} + \frac{1}{\nabla_1 u}\sum_{k\geq 2}\nabla_k u A^{1k} \\ \leq &-\frac{a}{\phi}|\nabla u|^2 + C(1 + |\nabla u| + |\nabla u|^{\gamma-2}) < 0 \end{aligned} \tag{5.24}$$

provided $|\nabla u|$ is sufficiently large. Therefore, by (5.16),

$$f_1 \geq \nu_0\Big(1 + \sum_{i=1}^{n} f_i\Big)$$

and a bound $|\nabla u(x_0,t_0)| \leq C$ follows from (5.23). □

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